TRACE RATIO MULTILINEAR DIMENSIONALY REDUCTION METHODS IN DATA SCIENCE

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Key words. Dimensionality Reduction; Multilinear Algebra; Trace-Ratio; Tensor Decompositions, t-product.

In this work, we propose an extension of trace ratio based Manifold learning methods to deal with multidimensional data sets. Based on recent progress on the tensor-tensor product, we present a generalization of the trace ratio criterion by using the properties of the t-product. We present a tensor Lanczos algorithm for solving the trace ratio problem. Manifold learning methods such as Laplacian eigenmaps, linear discriminant analysis and locally linear embedding will be formulated in a tensor representation and optimized by the proposed algorithm. We show the performance of the proposed dimension reduction method on several synthetic and real world data sets.

The trace ratio tensor problem is an important concept in machine learning for tasks such as feature extraction and dimensionality reduction, as it aids in the analysis of complicated, multidimensional data and can be expressed as follows

$$\max_{\mathscr{V}\in\mathbb{R}^{n_1\times d\times n_3}} \frac{\operatorname{Trace}\left[\mathscr{V}^T\star\mathscr{A}\star\mathscr{V}\right]}{\operatorname{Trace}\left[\mathscr{V}^T\star\mathscr{B}\star\mathscr{V}\right]},\ \text{subject to}\ \ \mathscr{V}^T\star\mathscr{V}=\mathscr{I}_d, \tag{0.1}$$

where $\mathscr{V}$, $\mathscr{A}$ and $\mathscr{V}$ are tensors. In this formulation, $\mathscr{V}\in\mathbb{R}^{n\times d\times n_3}$ is required to have f-orthonormal lateral slices, i.e., $\mathscr{V}(:,i,:)^T\star\mathscr{V}(:,i,:)=\boldsymbol{e}$, with $\boldsymbol{e}\in\mathbb{R}^{1\times 1\times n_3}$ has zero components and $\boldsymbol{e}(1,1,1)=1$, and $\mathscr{A}\in\mathbb{R}^{n\times n\times n_3}$ is an f-symmetric tensor, $\mathscr{B}\in\mathbb{R}^{n\times n\times n_3}$ is assumed to be f-symmetric and positive definite tensor. This problem can be replaced by a simpler, yet not equivalent problem

$$\max_{\mathscr{V}\in\mathbb{R}^{n\times d\times n_3}} \operatorname{Trace}\left[\mathscr{V}^T\star\mathscr{A}\star\mathscr{V}\right],\ \text{subject to}\ \mathscr{V}^T\star\mathscr{B}\star\mathscr{V}=\mathscr{I}_d. \tag{0.2}$$

In practice, Problem (0.1) often arises as a simplification of an objective function that is more difficult to maximize, which can be described as follows

$$\max_{\mathscr{V}\in\mathbb{R}^{n\times d\times n_3}} \frac{\operatorname{Trace}\left[\mathscr{V}^T\star\mathscr{A}\star\mathscr{V}\right]}{\operatorname{Trace}\left[\mathscr{V}^T\star\mathscr{B}\star\mathscr{V}\right]},\ \text{subject to}\ \ \mathscr{V}^T\star\mathscr{C}\star\mathscr{V}=\mathscr{I}_d, \tag{0.3}$$

where $\mathscr{B}$ and $\mathscr{C}$ are assumed to be f-symmetric and positive definite for simplicity. The tensor $\mathscr{C}$ defines the desired f-orthogonality and in the simplest case, it is just the Identity tensor.
The main theorem is the following one

THEOREM 0.1. *Let $\mathscr{A}\in\mathbb{R}^{n\times n\times n_3}$ and $\mathscr{B}\in\mathbb{R}^{n\times n\times n_3}$, where $\mathscr{A}$ is f-symmetric and $\mathscr{B}$ is an f-symmetric positive definite tensor. Consider the optimization problem as follows*

$$\max_{\mathscr{U}\in\mathbb{R}^{n\times d\times n_3}} \operatorname{Trace}\left(\mathscr{U}^T\star\mathscr{A}\star\mathscr{U}\right),\ \textit{subject to}\ \mathscr{U}^T\star\mathscr{B}\star\mathscr{U}=\mathscr{I}_d. \tag{0.4}$$

The problem (0.4) *achieves a maximum, and the solution of* (0.4) *is the d eigenslices associated to the d largest eigentube of the following generalized eigentube problem*

$$\mathscr{A}\star\mathscr{U}=\mathscr{B}\star\mathscr{U}\star\Lambda, \tag{0.5}$$

where $\Lambda\in\mathbb{R}^{d\times d\times n_3}$ is a f-diagonal tensor.
In the case of minimization, the solution of (0.4) *is the d eigenslices associated with the d smallest non-zero eigentube of the generalized eigentube problem* (0.5).